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Folded Crossed Cube with Five or More Dimensions Is Not Vertex-Transitive

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***Abstract*—Kulasinghe and Bettayeb [Inform. Process. Lett. 53 (1995) 33-36] proved that the crossed cube CQ_n (a synonym called multiply twisted hypercube in that paper) fails to be vertex-transitive for $n \geqslant 5$. In this paper, we study vertex-transitivity on folded crossed cubes FCQ_n and show that FCQ_n is vertex-transitive if and only if $n \in \{1, 2, 4\}$.**

Keywords—Vertex-transitivity; Folded Crossed Cubes; Interconnection networks.

I. INTRODUCTION

Interconnection networks are usually modeled as undirected simple graphs $G = (V, E)$, where the vertex set V and the edge set E represent the sets of processing elements and communication channels, respectively. One of the central issues of an interconnection network is to consider the symmetry. Formally, an *automorphism* of a graph $G = (V, E)$ is a permutation ϕ on V such that the pair of vertices (u, v) form an edge in E if and only if the pair $(\phi(u), \phi(v))$ also form an edge in E. We say that two vertices v and w belong to the same *orbit* if there is an automorphism ϕ of G such that $\phi(v) = w$.

Definition 1. (See [4], [10].) A graph $G = (V, E)$ is *vertex-transitive* (also known as *vertex-symmetric*) if for every two vertices $v, w \in V$, there exists an automorphism of G which maps v to w. That is, a vertex-transitive graph has just one orbit.

Intuitively, a vertex-transitive graph looks the same when we take a view from every vertex. The vertex-transitive property is advantageous to the design and simulation of some algorithms in graphs. Akers and Krishnamurthy [3] showed that every Cayley graph over a general group is vertex-transitive, and thus is regular. Accordingly, the lack of vertex-transitivity for a class of graphs $\mathscr{C}$ does remove $\mathscr{C}$ from the family of Cayley graphs. For example, Abraham and Padmanabhan [1] and Cull and Larson [5] respectively pointed out that topologies for multiprocessor systems such as twisted cubes TQ_n and Möbius cubes MQ_n are possessed of the property of asymmetry. In addition, Kulasinghe and Bettayeb [11] showed that crossed cubes CQ_n fail to be vertex-transitive for $n \geqslant 5$. Liu et al. [12] found out that although locally twisted cubes LTQ_n are not vertex-transitive for $n \geqslant 4$, these cubes are in possession of the property of *even-odd-vertex-transitivity*, i.e., each of them has just two orbits for which every pair of vertices with the same parity belong to the same orbit.

In this paper, we investigate the vertex-transitivity of folded crossed cubes (defined later in Section II) and show the following result.

Theorem 2. *The folded crossed cube FCQ_n is vertex-transitive if and only if $n \in \{1, 2, 4\}$.*

II. FOLDED CROSSED CUBES

The n-dimensional crossed cube CQ_n, proposed first by Efe [6], [7], is a variant of an n-dimensional hypercube. One advantage of CQ_n is that the diameter is only about one half of the diameter of an n-dimensional hypercube. For more properties of CQ_n, the reader can refer to [8], [11]

In this paper, we use a unique binary string $v_{n-1}v_{n-2}\cdots v_1v_0$, which is also called a *lable*, of length n to identify a vertex v in CQ_n. For conciseness, sometimes labels of vertices are also represented by their decimal. Two 2-bit binary strings $v = v_1v_0$ and $w = w_1w_0$ are *pair-related*, denoted by $v \sim w$, if and only if $(v, w) \in \{(00, 00), (10, 10), (01, 11), (11, 01)\}$. CQ_n is the labeled graph with the following recursively fashion:

Definition 3. (See [6], [7].) CQ_1 is the complete graph on two vertices with labels 0 and 1. For $n \geqslant 2$, CQ_n consists of two subcubes CQ^0_{n-1} and CQ^1_{n-1} such that every vertex in CQ^0_{n-1} and CQ^1_{n-1} is labeled by 0 and 1 in its leftmost bit, respectively. Two vertices $v = 0v_{n-2}\cdots v_1v_0 \in V(CQ^0_{n-1})$ and $w = 1w_{n-2}\cdots w_1w_0 \in V(CQ^1_{n-1})$ are joined by an edge if and only if

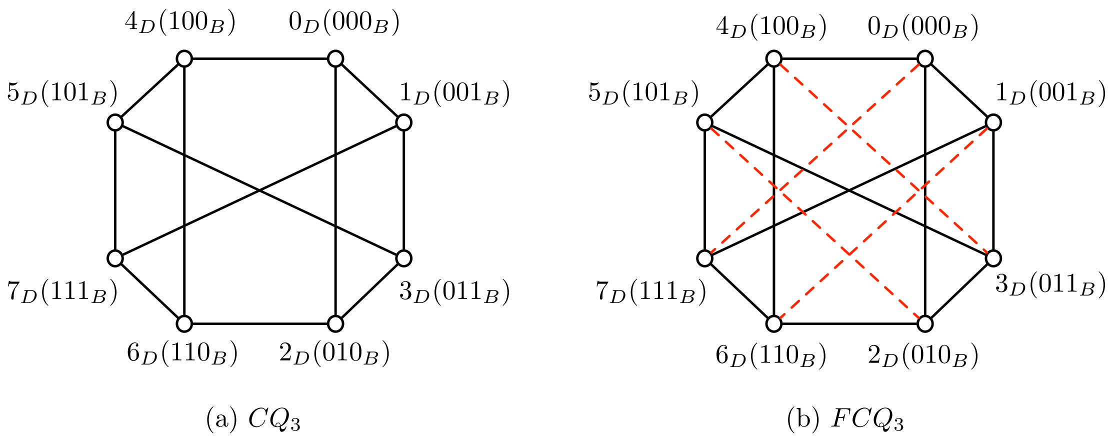

Fig. 1. Crossed cube CQ_3 and folded crossed cube FCQ_3, where v_D (respectively, v_B) denotes the decimal (respectively, binary) representation of the vertex v.

(1) $v_{n-2} = w_{n-2}$ if n is even, and

(2) $v_{2i+1}v_{2i} \sim w_{2i+1}w_{2i}$ for $0 \leqslant i < \lfloor (n-1)/2 \rfloor$.

Let $\mathbb{Z}_n = \{0, 1, \ldots, n-1\}$. Crossed cubes can also be defined equivalently as follows:

Lemma 4. (See [6], [7].) *Two vertices $v = v_{n-1}v_{n-2}\cdots v_0$ and $w = w_{n-1}w_{n-2}\cdots w_0$ are joined by an edge in CQ_n if and only if there exists an integer $i \in \mathbb{Z}_n$ such that*

(1) $v_{n-1}v_{n-2}\cdots v_{i+1} = w_{n-1}w_{n-2}\cdots w_{i+1}$,

(2) $v_i \neq w_i$,

(3) $v_{i-1} = w_{i-1}$ *if i is odd, and*

(4) $v_{2j+1}v_{2j} \sim w_{2j+1}w_{2j}$ *for $i \in \mathbb{Z}_{\lfloor i/2 \rfloor}$.*

In the above lemma, v and w have the leftmost differing bit at position i. In this case, v and w are said to be the *i-neighbors* to each other, and for notational convenience we write $w = N_i(v)$ or $v = N_i(w)$. Also, the edge (v, w) is called an *i-dimensional edge* of CQ_n.

Inspired by the idea of El-Amawy and Latifi [9] that proposed the so-called folded hypercubes to strengthen the structure of hypercubes, a variation of crossed cubes was first introduced in [13] as follows. The *n-dimensional folded crossed cube*, denoted FCQ_n, is constructed from CQ_n by adding a set of edges (v, w) with $v_{n-1}v_{n-2}\cdots v_1v_0 = \bar{w}_{n-1}\bar{w}_{n-2}\cdots \bar{w}_1\bar{w}_0$ for $0 \leqslant v \leqslant (2^n - 1)$. Hence, every edge in such a set is called a *complement edge*. Similarly, we use $w = N_*(v)$ or $v = N_*(w)$ to denote the adjacency of v and w in this case. Fig. 3 shows crossed cube CQ_3 and folded crossed cube FCQ_3, where solid lines indicate normal edges and dashed lines represent complement edges. Refer to [2] for more properties of FCQ_n.

Throughout this paper, we also use the following notations. Let $N_G(v)$ denote the set of vertices adjacent to v in a graph G, and we omit the subscript G if it is clear from the context. We use $P_n = v_0 \to v_1 \to \cdots \to v_n$ to stand for a path of length n starting from v_0. Moreover, $P_n = v_0 \xrightarrow{d_1} v_1 \xrightarrow{d_2} \cdots \xrightarrow{d_n} v_n$ indicates that vertices v_{i-1} and v_i are connected by a d_i-dimensional edge or a complement edge if $d_i =$ "$*$". In particular, the notation P_n^w means that the path P_n does not pass through a vertex w.

III. Proof of the Main Result

Since FCQ_1 and FCQ_2 are isomorphic to complete graph, respectively, with two vertices and four vertices (i.e., K_2 and K_4), the following lemma is an immediate result.

Lemma 5. *FCQ_1 and FCQ_2 are vertex-transitive.*

Lemma 6. *FCQ_3 is not vertex-transitive.*

Proof. Suppose on the contrary that FCQ_3 is vertex-transitive, i.e., there exists an automorphism ϕ of FCQ_3 such that $\phi(1) = 0$. From Figure 3(b), we can easily check $N(1) = \{0, 3, 6, 7\}$ and $N(0) = \{1, 2, 4, 7\}$. Let G_i be the subgraph of FCQ_3 induced by $N(i)$ for $i = 0, 1$. Clearly, G_1 contains a path of length 2 (i.e., the path $0 \to 7 \to 6$), whereas G_0 does not. This contradicts that ϕ is an automorphism of FCQ_3. □

Lemma 7. *FCQ_4 is vertex-transitive.*

Proof. Let e be the identity permutation on the vertices of FCQ_4, i.e., $e(i) = i$ for all $i \in \mathbb{Z}_{16}$. If ϕ is an automorphism of G, then so is its inverse ϕ^{-1}, and if ψ is a second automorphism of G, then the product $\phi \cdot \psi$ is an automorphism. Hence, to show the result, we only need to provide automorphisms ϕ_i, $i = 1, 2, \ldots, 15$, such that $\phi_i(0) = i$ (i.e., $\phi_i^{-1}\phi_j$ maps vertex i to vertex j for any $i, j \in \mathbb{Z}_{16}$). The following are the desired automorphism:

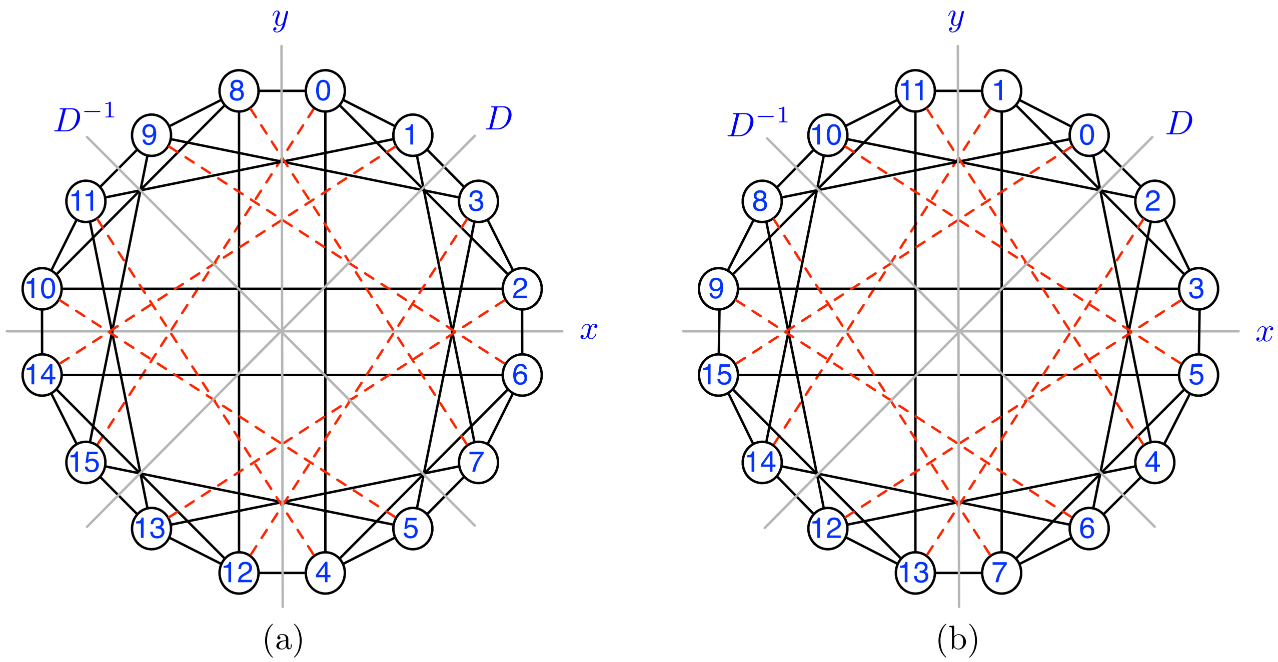

Fig. 2. Illustration of automorphisms of FCQ_4.

$\phi_1 = (1, 0, 3, 2, 7, 6, 5, 4, 11, 10, 9, 8, 13, 12, 15, 14)$

(see Fig. 2(a) and 2(b) for illustration);

$\phi_2 = (2, 3, 0, 1, 10, 11, 8, 9, 6, 7, 4, 5, 14, 15, 12, 13)$

(see Fig. 2(a) and its flip on the diagonal D);

$\phi_3 = (3, 2, 1, 0, 9, 8, 11, 10, 5, 4, 7, 6, 15, 14, 13, 12)$

(apply ϕ_1, see Fig. 2(b) and then flip on the diagonal D);

$\phi_4 = (4, 5, 6, 7, 0, 1, 2, 3, 12, 13, 14, 15, 8, 9, 10, 11)$

(see Fig. 2(a) and its flip on the x-axis);

$\phi_5 = (5, 4, 7, 6, 15, 14, 13, 12, 3, 2, 1, 0, 9, 8, 11, 10)$

(apply ϕ_1, see Fig. 2(b) and then rotate 90 degrees in clockwise direction);

$\phi_6 = (6, 7, 4, 5, 2, 3, 0, 1, 14, 15, 12, 13, 10, 11, 8, 9)$

(see Fig. 2(a) and its rotation of 90 degrees in clockwise direction);

$\phi_7 = (7, 6, 5, 4, 1, 0, 3, 2, 13, 12, 15, 14, 11, 10, 9, 8)$

(apply ϕ_1, see Fig. 2(b) and then flip on the x-axis);

$\phi_8 = (8, 9, 10, 11, 12, 13, 14, 15, 0, 1, 2, 3, 4, 5, 6, 7)$

(see Fig. 2(a) and its flip on the y-axis);

$\phi_9 = (9, 8, 11, 10, 3, 2, 1, 0, 15, 14, 13, 12, 5, 4, 7, 6)$

(apply ϕ_1, see Fig. 2(b) and then rotate 90 degrees in counterclockwise direction);

$\phi_{10} = (10, 11, 8, 9, 2, 3, 0, 1, 14, 15, 12, 13, 6, 7, 4, 5)$

(see Fig. 2(a) and its rotation of 90 degrees in counterclockwise direction);

$\phi_{11} = (11, 10, 9, 8, 13, 12, 15, 14, 1, 0, 3, 2, 7, 6, 5, 4)$

(apply ϕ_1, see Fig. 2(b) and then flip on the y-axis);

$\phi_{12} = (12, 13, 14, 15, 8, 9, 10, 11, 4, 5, 6, 7, 0, 1, 2, 3)$

(see Fig. 2(a) and its flip on both x-axis and y-axis);

$\phi_{13} = (13, 12, 15, 14, 11, 10, 9, 8, 7, 6, 5, 4, 1, 0, 3, 2)$

(apply ϕ_1, see Fig. 2(b) and then flip on both x-axis and y-axis);

$\phi_{14} = (14, 15, 12, 13, 6, 7, 4, 5, 10, 11, 8, 9, 2, 3, 0, 1)$

(see Fig. 2(a) and its flip on the anti-diagonal D^{-1});

$\phi_{15} = (15, 14, 13, 12, 5, 4, 7, 6, 9, 8, 11, 10, 3, 2, 1, 0)$

(apply ϕ_1, see Fig. 2(b) and then flip on the anti-diagonal D^{-1}). $\square$

Lemma 8. *FCQ_5 is not vertex-transitive.*

Proof. Suppose on the contrary that FCQ_5 is vertex-transitive, i.e., there exists an automorphism ϕ of FCQ_5 such that $\phi(0) = 4$. By Lemma 4 and the complement edges, we have $N(0) = \{1, 2, 4, 8, 16, 31\}$ and $N(4) = \{0, 5, 6, 12, 27, 28\}$. We prove the lemma through the following two claims.

Claim 1. *Let $w \in N(4)$ be any vertex. There exist at least three vertices $v_1, v_2, v_3 \in N(4)\setminus\{w\}$ such that v_i is connected by a P_2^4 starting from w in FCQ_5 for $i \in \{1, 2, 3\}$.*

We directly expatiate on these paths as follows:

For $w = 0$, we have $0 \xrightarrow{*} 31 \xrightarrow{4} 5$, $0 \xrightarrow{1} 2 \xrightarrow{2} 6$ and $0 \xrightarrow{3} 8 \xrightarrow{2} 12$.

For $w = 5$, we have $5 \xrightarrow{4} 31 \xrightarrow{*} 0$, $5 \xrightarrow{*} 26 \xrightarrow{0} 27$ and $5 \xrightarrow{2} 3 \xrightarrow{*} 28$.

For $w = 6$, we have $6 \xrightarrow{2} 2 \xrightarrow{1} 0$, $6 \xrightarrow{*} 25 \xrightarrow{1} 27$ and $6 \xrightarrow{3} 14 \xrightarrow{1} 12$.

For $w = 12$, we have $12 \xrightarrow{2} 8 \xrightarrow{3} 0$, $12 \xrightarrow{1} 14 \xrightarrow{3} 6$ and $12 \xrightarrow{4} 20 \xrightarrow{3} 28$.

For $w = 27$, we have $27 \xrightarrow{0} 26 \xrightarrow{*} 5$, $27 \xrightarrow{1} 25 \xrightarrow{*} 6$ and $27 \xrightarrow{2} 29 \xrightarrow{0} 28$.

For $w = 28$, we have $28 \xrightarrow{*} 3 \xrightarrow{2} 5$, $28 \xrightarrow{3} 20 \xrightarrow{4} 12$ and $28 \xrightarrow{0} 29 \xrightarrow{2} 27$.

Claim 2. *Let $w = 1 \in N(0)$. There exist at most two vertices $v_1, v_2 \in N(0) \setminus \{1\}$ such that v_i is connected by a P_2^0 starting from w in FCQ_5 for $i \in \{1, 2\}$.*

By Lemma 4, we observe that

$$N(1) \setminus \{0\} = \{3, 7, 11, 19, 30\}.$$

Moreover,

$$N(3) \setminus \{1\} = \{2, 5, 9, 17, 28\},$$

$$N(7) \setminus \{1\} = \{5, 6, 13, 24, 29\},$$

$$N(11) \setminus \{1\} = \{9, 10, 13, 20, 25\},$$

$$N(19) \setminus \{1\} = \{12, 17, 18, 21, 25\},$$

and

$$N(30) \setminus \{1\} = \{6, 22, 26, 28, 31\}.$$

Thus, we can check that only the two vertices $2, 31 \in N(0) \setminus \{1\}$ are connected by a P_2^0 starting from w, i.e., $1 \xrightarrow{1} 3 \xrightarrow{0} 2$ and $1 \xrightarrow{*} 30 \xrightarrow{0} 31$.

According to the two claims, it contradicts that ϕ is an automorphism of FCQ_5. □

In what follows, we will consider the vertex-transitivity on FCQ_n with higher dimension, i.e., $n \geqslant 6$. Before this, we need some auxiliary properties.

Lemma 9. (See Lemma 5 in [11].) *For $n \geqslant 6$, there exists at most one pair of neighbors of vertex 0 in CQ_n that are not linked by a P_3.*

By Lemma 4, the following two properties can be obtained directly from the adjacency of vertices in FCQ_n.

Proposition 10. *For $v, w \in V(FCQ_n)$, if $v_{2k+1}v_{2k} = 01$ and $w_{2k+1}w_{2k} = 10$ or $v_{2k+1}v_{2k} = 00$ and $w_{2k+1}w_{2k} = 11$ for some $k \in \mathbb{Z}_{\lfloor n/2 \rfloor}$, then v and w cannot be adjacent with the exception of (v, w) being a complement edge.*

Proposition 11. *For $v, w \in V(FCQ_n)$, if $v_{2k+1}v_{2k} = w_{2k+1}w_{2k} = 01$ or $v_{2k+1}v_{2k} = w_{2k+1}w_{2k} = 11$ for some $k \in \mathbb{Z}_{\lfloor n/2 \rfloor}$, then v and w cannot be adjacent via an edge with dimension i for $i \geqslant 2k$ or a complement edge.*

Lemma 12. *For $n \geqslant 6$, every vertex $v \in N(0) \setminus \{1, 2^n - 1\}$ in FCQ_n is connected by a P_3 starting from $2^n - 1$.*

Proof. Clearly, $v \in \{2^i\colon 1 \leqslant i \leqslant n\}$. We expatiate on these paths as follows. If $v = 2^1$, we have

$$(2^n - 1) \xrightarrow{2} (2^n - 2^2 - 2^1 - 1) \xrightarrow{*} (2^2 + 2^1) \xrightarrow{2} 2^1.$$

If $v = 2^i$ for even $i \geqslant 2$, we have

$$(2^n - 1) \xrightarrow{i} (2^n - 2^i - \sum_{j=1}^{i/2} 2^{2j-1} - 1) \xrightarrow{i-1} (2^n - 2^i - 1) \xrightarrow{*} 2^i.$$

If $v = 2^i$ for odd $i \geqslant 3$, we have

$$(2^n - 1) \xrightarrow{i} (2^n - 2^i - \sum_{j=1}^{\lfloor i/2 \rfloor} 2^{2j-1} - 1) \xrightarrow{i-2} (2^n - 2^i - 1) \xrightarrow{*} 2^i.$$

□

For example, we consider some vertices

$$v \in N(0) \setminus \{1, 2^7 - 1\}$$

in FCQ_7. If $v = 2$, we have

$$1111111_B \xrightarrow{2} 1111001_B \xrightarrow{*} 0000110_B \xrightarrow{2} 0000010_B = v.$$

If $v = 2^6$, we have

$$1111111_B \xrightarrow{6} 0010101_B \xrightarrow{5} 0111111_B \xrightarrow{*} 1000000_B = v.$$

If $v = 2^5$, we have

$$1111111_B \xrightarrow{5} 1010101_B \xrightarrow{3} 1011111_B \xrightarrow{*} 0100000_B = v.$$

Lemma 13. *For $n \geqslant 6$, there exists at most two pairs of neighbors of vertex 0 in FCQ_n that are not linked by a P_3.*

Proof. Clearly, $N_{FCQ_n}(0) = N_{CQ_n}(0) \cup \{2^n - 1\}$. Lemma 12 shows that there exists a P_3 between vertices $2^n - 1$ and v in FCQ_n for every $v \in N_{FCQ_n}(0) \setminus \{1, 2^n - 1\}$. Since Lemma 9 has already shown that there exists at most one pair of vertices $u, v \in N_{CQ_n}(0)$ $(= N_{FCQ_n}(0) \setminus \{2^n - 1\})$ without linking by a P_3, the result directly follows no matter whether there is a P_3 or not between vertices 1 and $2^n - 1$ in FCQ_n. □

Lemma 14. *For $n \geqslant 6$, there exists at least three pairs of neighbors of vertex 1 in FCQ_n that are not linked by a P_3.*

Proof. Since $n \geqslant 6$, by Lemma 4, we have $N_0(1) = 0$, $N_2(1) = 7$, $N_3(1) = 11$, $N_4(1) = 19$, $N_5(1) = 35$ and $N_*(1) = 2^n - 2$. We claim that each of the pairs $(7, 11)$, $(19, 35)$ and $(0, 2^n - 2)$ does not be linked by a P_3. For the pair $(7, 11)$, we suppose on the contrary that FCQ_n contains the following path:

$$v(= 0 \cdots 000111_B = 7_D) \xrightarrow{\alpha} x \xrightarrow{\beta} y \xrightarrow{\gamma}$$
$$w(= 0 \cdots 001011_B = 11_D).$$

Note that $\alpha \neq \beta$ and $\beta \neq \gamma$. We consider the following cases.

Case 1: $\beta =$ "$*$". In this case, $\alpha, \gamma \in \mathbb{Z}_n$. Since $x = N_\alpha(v)$ and $y = N_\gamma(w)$, if $\alpha, \gamma \geqslant 1$ or $\alpha = \gamma = 0$, then x and y have the same parity, and thus this contradicts that (x, y) is a complement edge. For $\alpha > \gamma = 0$, since $w_5w_4 = 00$, it implies $y_5y_4 = 00$ and $x_5x_4 = 11$. Since $v_5v_4 = 00$ and $\alpha \neq \beta$, by Proposition 10, x and v are nonadjacent, a contradiction. A similar argument shows that the condition $\gamma > \alpha = 0$ also leads to a contradiction.

Case 2: $\beta = 0$. Clearly, either $x_0 = \bar{y}_0 = 0$ or $y_0 = \bar{x}_0 = 0$. Without loss of generality, we assume $x_0 = 0$ and $y_0 = 1$. Since $v_0 = 1$ and $x = N_\alpha(v)$, either $\alpha = 0$ or $\alpha =$ "$*$". Since $\alpha \neq \beta$, we only need to consider $\alpha =$ "$*$". Thus, $v_5v_4 = 00$ implies $x_5x_4 = y_5y_4 = 11$. Also, since $w_5w_4 = 00$ and $N_\gamma(y) = w$, by Proposition 10, (y, w) must be a complement edge (i.e., $\gamma =$ "$*$"). However, this contradicts the fact that $y_0 = w_0 = 1$.

Case 3: $\beta \in \mathbb{Z}_n \setminus \{0\}$. Clearly, either $x_0 = y_0 = 0$ or $x_0 = y_0 = 1$. We first consider $x_0 = y_0 = 0$. In this case, we have $\alpha, \gamma \in \{0, \text{“}*\text{”}\}$. If $\alpha = \gamma = 0$ (respectively, $\alpha = \gamma =$ "$*$"), then $x_3x_2 = 01$ and $y_3y_2 = 10$ (respectively, $x_3x_2 = 10$ and $y_3y_2 = 01$). If $\alpha = 0$ and $\gamma =$ "$*$" (respectively, $\alpha =$ "$*$" and $\gamma = 0$), then $x_5x_4 = 00$ and $y_5y_4 = 11$ (respectively, $x_5x_4 = 11$ and $y_5y_4 = 00$). Since (x, y) is not a complement edge, by Proposition 10, all of the above situations imply that x and y are nonadjacent, a contradiction. Next, we consider $x_0 = y_0 = 1$. Since $v_0 = w_0 = 1$, we have $\alpha, \gamma \geqslant 1$, and it further implies that $x_1x_0 = y_1y_0 = 01$. Since (x, y) is not a complement edge, by Proposition 11, x and y are nonadjacent. This again leads to a contradiction.

For the pair $(19, 35)$, we let $v = 0 \cdots 010011_B = 19_D$ and $w = 0 \cdots 100011_B = 35_D$. To show that v and w are not linked by a P_3, the proof is similar to the above argument by dealing with the second and the third bits of the labels of vertices (e.g. v_3v_2 and w_3w_2) instead of the fourth and the fifth bits (e.g. v_5v_4 and w_5w_4), and vice versa.

For the pair $(0, 2^n - 2)$, we suppose on the contrary that FCQ_n contains the following path:

$$v(= 0 \cdots 000000_B = 0_D) \xrightarrow{\alpha} x \xrightarrow{\beta} y \xrightarrow{\gamma}$$
$$w(= 1 \cdots 111110_B = 2^n - 2).$$

Note that $\alpha \neq \beta$ and $\beta \neq \gamma$. We first observe that a 0-dimensional edge cannot occur immediately before or after a complement edge because $(0 \cdots 000000_B = 0_D) \xrightarrow{0} (0 \cdots 000001_B = 1_D) \xrightarrow{*} (1 \cdots 111110_B = 2^n - 2)$ and $(0 \cdots 000000_B = 0_D) \xrightarrow{*} (1 \cdots 111111_B = 2^n - 1) \xrightarrow{0} (1 \cdots 111110_B = 2^n - 2)$ are paths of length two that connect v and w in FCQ_n. Moreover, we have $\alpha, \gamma \notin \{0, \text{“}*\text{”}\}$. If $\beta =$ "$*$", we have $\alpha, \gamma \geqslant 1$. Since $x = N_\alpha(v)$ and $y = N_\gamma(w)$, x and y have the same parity, a contradiction. If $\beta = 0$, then either $x_0 = 1$ or $y_0 = 1$, which implies that one of α and γ must be contained in the set $\{0, \text{“}*\text{”}\}$, a contradiction. Finally, we consider $\beta \in \mathbb{Z}_n \setminus \{0\}$. Since $\alpha, \gamma \geqslant 1$ and $\alpha \neq \beta$, the label of vertex y contains exactly two "1"s, which implies that y cannot be adjacent to w in FCQ_n, a contradiction. □

Lemma 15. *FCQ_n is not vertex-transitive for $n \geqslant 6$.*

Proof. This is an immediate result of Lemma 13 and Lemma 14. □

According to Lemmas 5, 6, 7, 8 and 15, we complete the proof of Theorem 2.

IV. Concluding Remarks

An open question arises from this paper is as follows. Let $G = (V, E)$ be a graph and define a binary relation $\mathscr{R} = \{(u, v) \in V \times V \colon u \text{ and } v \text{ have the same orbit}\}$. Obviously, $\mathscr{R}$ is an equivalence relation on V. Let $Orb(G)$ denote the number of orbits inhabited in a graph G (i.e., the number of equivalence classes of V by $\mathscr{R}$). As we have mentioned earlier, Liu et al. [12] showed that locally twisted cubes LTQ_n always possess two orbits (i.e., $Orb(LTQ_n) = 2$) for $n \geqslant 4$. In addition, Kulasinghe and Bettayeb [11] showed that $Orb(CQ_n) \neq 1$ if $n \geqslant 5$. In this paper, we prove that $Orb(FCQ_n) = 1$ if and only if $n \in \{1, 2, 4\}$. It would be an interesting question to determine $Orb(CQ_n)$ or $Orb(FCQ_n)$ for arbitrary n.

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